Calculus 2
Final Exam – Solutions
Exam Date: April 11, 2022 (16:00 – 18:00)

university of groningen

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1) Consider the hyperboloid of one sheet H given by the equation

$$x^2 + \frac{y^2}{9} - \frac{z^2}{4} = 2$$

[8] a) Treating H as a level surface of a function of three variables, find an equation of the tangent plane to H at the point $P(3, 9, 8)$.

[8] b) Use the Implicit Function Theorem to show that near the point P in part a), H can be considered to be the graph of a function f of x and z. Compute the partial derivatives f_x and f_z and show that the tangent plane found in a) coincides with the graph of the linearization $L(x, z)$ of $f(x, z)$ at $(3, 8)$.

[8] c) Use the method of Lagrange multipliers to find the point $Q(x_*, y_*, z_*)$ on the tangent plane in part a) that is closest to the origin. Determine the distance of between the tangent plane and the origin.

Solution. a) The hyperboloid H can be viewed as a level surface for the function $F(x, y, z) = x^2 + \frac{y^2}{9} - \frac{z^2}{4}$. To obtain a normal vector to the tangent plane we may use the fact that the gradient vector is perpendicular to level surfaces. The gradient vector of $F(x, y, z)$ is

$$\nabla F(x, y, z) = F_x\,\vec{\imath} + F_y\,\vec{\jmath} + F_z\,\vec{k} = 2x\,\vec{\imath} + \tfrac{2}{9}y\,\vec{\jmath} - \tfrac{1}{2}z\,\vec{k}$$

which at the point $(3, 9, 8)$ becomes

$$\vec{n} = \nabla F(3, 9, 8) = 2(3)\,\vec{\imath} + \tfrac{2}{9}(9)\,\vec{\jmath} - \tfrac{1}{2}(8)\,\vec{k} = 6\,\vec{\imath} + 2\,\vec{\jmath} - 4\,\vec{k}.$$

For any point $Q(x, y, z)$ in the tangent plane, the vector $\overrightarrow{PQ} = (x-3)\,\vec{\imath} + (y-9)\,\vec{\jmath} + (z-8)\,\vec{k}$ lies in the plane and as such it is perpendicular to $\vec{n}$, i.e. we have

$$\vec{n} \cdot \overrightarrow{PQ} = 0 \quad \Leftrightarrow \quad 6(x-3) + 2(y-9) - 4(z-8) = 0 \quad \Leftrightarrow \quad 6x + 2y - 4z - 4 = 0.$$

Therefore $3x + y - 2z - 2 = 0$ is an equation for the tangent plane to H at $(3, 9, 8)$.

b) Since $F(x, y, z)$ is a polynomial function of x, y, z, its partial derivatives are continuous. Furthermore, we have $F(3, 9, 8) = 2$ and $F_y(3, 9, 8) = \frac{2}{9}y|_{y=9} = 2 \neq 0$. By the Implicit Function Theorem, there is a neighbourhood of $(3, 9, 8)$ in which a unique function $y = f(x, z)$ is defined and satisfies $F(x, f(x, z), z) = 2$. The partial derivatives of f are found via implicit differentiation

$$f_x = -\frac{F_x}{F_y} = -\frac{2x}{\frac{2}{9}y} = -\frac{9x}{y}, \qquad f_z = -\frac{F_z}{F_y} = -\frac{-\frac{1}{2}z}{\frac{2}{9}y} = \frac{9z}{4y}$$

taking the following values at $(3, 9, 8)$:

$$f_x(3, 8) = -\frac{9(3)}{9} = -3, \qquad f_z(3, 8) = \frac{9(8)}{4(9)} = 2.$$

Hence the linearization of $y = f(x, z)$ at $(3, 8)$ is

$$\begin{aligned} L(x, z) &= f_x(3, 8)(x-3) + f_z(3, 8)(z-8) + f(3, 8) \\ &= -3(x-3) + 2(z-8) + 9 \\ &= -3x + 2z + 2. \end{aligned}$$

The graph of the linearization is given by the equation $y = -3x+2z+2$ which coincides with the equation of the tangent plane found in part a).

c) To find the point $Q(x_*, y_*, z_*)$ is to minimize the distance from the origin which is equivalent to minimizing the function $d(x, y, z) = x^2 + y^2 + z^2$ (i.e. distance from the origin squared) subject to the constraint $g(x, y, z) = 3x + y - 2z = 2$. We use the method of Lagrange multipliers and solve

$$\begin{cases} \nabla d(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 2x = 3\lambda \\ 2y = \lambda \\ 2z = -2\lambda \\ 3x + y - 2z = 2 \end{cases}$$

for x, y, z, λ. From the first three equations we get $x = \frac{3}{2}\lambda$, $y = \frac{1}{2}\lambda$, $z = -\lambda$, which plugged into the last equation yields $7\lambda = 2$, hence $\lambda = \frac{2}{7}$. Therefore we find the coordinates of the point Q to be $x_* = \frac{3}{2}(\frac{2}{7}) = \frac{3}{7}$, $y_* = \frac{1}{2}(\frac{2}{7}) = \frac{1}{7}$, $z_* = -\frac{2}{7}$, i.e. $Q(x_*, y_*, z_*) = (\frac{3}{7}, \frac{1}{7}, -\frac{2}{7})$. Its distance from the origin is

$$D = |OQ| = \sqrt{d(x_*, y_*, z_*)} = \frac{1}{7}\sqrt{3^2 + 1^2 + (-2)^2} = \frac{\sqrt{14}}{7} = \sqrt{\frac{2}{7}}.$$

Alternatively, this distance can be found computing the length of the projection of the vector $\overrightarrow{OP}$ to $\vec{n}$:

$$D = \frac{|\vec{n} \cdot \overrightarrow{OP}|}{|\vec{n}|} = \frac{|6(3) + 2(9) - 4(8)|}{\sqrt{6^2 + 2^2 + (-4)^2}} = \frac{4}{\sqrt{56}} = \frac{2}{\sqrt{14}} = \sqrt{\frac{2}{7}}.$$

2) Consider the vector field

$$\vec{G}(x, y, z) = \frac{Ax}{x^2 + y^2 + 1}\,\vec{\imath} + \left(\frac{2y}{x^2 + y^2 + 1} + Bze^y\right)\vec{\jmath} + e^y\,\vec{k}$$

with parameters $A, B \in \mathbb{R}$.

9 a) Determine the values of A and B for which $\vec{G}$ is conservative.

8 b) For A and B found in part a), determine a scalar potential for $\vec{G}$.

4 c) For A and B found in part a), compute the line integral of $\vec{G}$ along the curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $y = 1$ from the point $P_0(0, 1, 1)$ to the point $P_1(1, 1, 2)$.

Solution. a) Since $\vec{G}$ is defined everywhere on $\mathbb{R}^3$ and has continuously differentiable components, it is conservative if and only if $\operatorname{curl}\vec{G} = \vec{0}$. We have

$$\operatorname{curl}\vec{G} = \nabla \times \vec{G} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{Ax}{x^2+y^2+1} & \frac{2y}{x^2+y^2+1} + Bze^y & e^y \end{vmatrix}$$

$$\begin{aligned}
= \quad & \left[\frac{\partial}{\partial y}(e^y) - \frac{\partial}{\partial z}\left(\frac{2y}{x^2+y^2+1} + Bze^y\right)\right]\vec{\imath} \\
& + \left[\frac{\partial}{\partial z}\left(\frac{Ax}{x^2+y^2+1}\right) - \frac{\partial}{\partial x}(e^y)\right]\vec{\jmath} \\
& + \left[\frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2+1} + Bze^y\right) - \frac{\partial}{\partial y}\left(\frac{Ax}{x^2+y^2+1}\right)\right]\vec{k} \\
= & \left[e^y - Be^y\right]\vec{\imath} + [0-0]\,\vec{\jmath} + \left[\frac{-4xy}{(x^2+y^2+1)^2} - \frac{-2Axy}{(x^2+y^2+1)^2}\right]\vec{k} \\
= & e^y\,[1-B]\,\vec{\imath} + \frac{-2xy}{(x^2+y^2+1)^2}\,[2-A]\,\vec{k}
\end{aligned}$$

so $\operatorname{curl}\vec{G} = \vec{0}$ everywhere if and only if $A = 2$ and $B = 1$:

$$\vec{G}(x,y,z) = \frac{2x}{x^2+y^2+1}\vec{\imath} + \left(\frac{2y}{x^2+y^2+1} + ze^y\right)\vec{\jmath} + e^y\,\vec{k}.$$

b) We need to solve the equation $\nabla g = \vec{G}$ with $A = 2$, $B = 1$ for g. Written in component form, it reads

$$g_x = \frac{2x}{x^2+y^2+1}, \tag{1}$$

$$g_y = \frac{2y}{x^2+y^2+1} + ze^y, \tag{2}$$

$$g_z = e^y. \tag{3}$$

Integrating both sides of eq. (1) with respect to x, we obtain

$$g(x,y,z) = \ln(x^2+y^2+1) + h(y,z), \tag{4}$$

where $h(y,z)$ is a constant of integration depending on y and z (but not x). Differentiating (4) with respect to y, we get

$$g_y = \frac{2y}{x^2+y^2+1} + h_y(y,z) \tag{5}$$

Comparing eqs. (2) and (5) gives

$$h_y(y,z) = ze^y \tag{6}$$

which integrated with respect to y yields

$$h(y,z) = ze^y + k(z). \tag{7}$$

Again, we have constant of integration $k(z)$ that may depend on z (but not y). Plugging this into eq. (4) gives us

$$g(x,y,z) = \ln(x^2+y^2+1) + ze^y + k(z). \tag{8}$$

Differentiating (8) with respect to z and comparing the result to (3) yields

$$k'(z) = 0 \quad \Rightarrow \quad k(z) = K \text{ (constant)}. \tag{9}$$

Therefore we find that

$$g(x, y, z) = \ln(x^2 + y^2 + 1) + ze^y + K \tag{10}$$

is a potential of the conservative vector field $\vec{G}$ (with $A = 2,\ B = 1$).

c) Since $\vec{G}$ (with $A = 2,\ B = 1$) is conservative, i.e. we have $\vec{G}(x, y, z) = \nabla g(x, y, z)$ with the potential $g(x, y, z) = \ln(x^2 + y^2 + 1) + ze^y + K$ we have

$$\int_{P_0 \to P_1} \vec{G} \cdot d\vec{r} = g(P_1) - g(P_0) = g(1, 1, 2) - g(0, 1, 1) = (\ln 3 + 2e) - (\ln 2 + e) = e + \ln 3 - \ln 2$$

by the Fundamental Theorem of line integrals.

Alternatively, we could parametrize the curve in question by the vector function

$$\vec{r}(x) = x\,\vec{\imath} + \vec{\jmath} + (x^2 + 1)\,\vec{k}, \qquad 0 \le x \le 1$$

whose derivative is

$$\vec{r}\,'(x) = \vec{\imath} + 2x\,\vec{k}, \qquad 0 \le x \le 1$$

and along which the vector fields takes the values

$$\vec{G}(\vec{r}(x)) = \frac{2x}{x^2 + 2}\,\vec{\imath} + \left(\frac{2}{x^2 + 2} + (x^2 + 1)e\right)\vec{\jmath} + e\,\vec{k}.$$

Hence the line integral can also be directly evaluated as follows

$$\int_{P_0 \to P_1} \vec{G} \cdot d\vec{r} = \int_0^1 \vec{G}(\vec{r}(x)) \cdot \vec{r}\,'(x)\,dx = \int_0^1 \left(\frac{2x}{x^2 + 2} + 2xe\right) dx = \left[\ln(x^2 + 2) + x^2 e\right]_{x=0}^{x=1} = \ln 3 + e - \ln 2.$$

3) Consider a fluid with the velocity field

$$\vec{V}(x, y, z) = \frac{-y}{\sqrt{x^2 + y^2}}\,\vec{\imath} + \frac{x}{\sqrt{x^2 + y^2}}\,\vec{\jmath} + (x^2 + y^2)\,z\,\vec{k}$$

and the surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ -y - 3 \le z \le y + 3\}$ with outward normal vectors and positively-oriented boundary ∂S.

3 a) Describe and sketch the surface S and its boundary ∂S (draw orientation).

Verify Stokes' Theorem by

8 b) calculating the circulation of $\vec{V}$ along ∂S, i.e. $\int_{\partial S} \vec{V} \cdot d\vec{r}$ and

12 c) computing the flux of $\operatorname{curl} \vec{V}$ across S, that is $\iint_S \operatorname{curl} \vec{V} \cdot d\vec{S}$.

Solution. a) The surface S is the part of the cylinder of radius 1 with the z-axis as axis that is above the plane $y + z + 3 = 0$ and below the plane $y - z + 3 = 0$. The boundary ∂S consists of the two ellipses C_1 and C_2 obtained by slicing the cylinder with each of the planes (see Figure 1):

$$C_1 = \{(x, y, z) \mid x^2 + y^2 = 1,\ z = y + 3\}, \quad C_2 = \{(x, y, z) \mid x^2 + y^2 = 1,\ z = -y - 3\}.$$

These ellipses can be thought of as the projection of the unit circle $x^2 + y^2 = 1$ onto the planes $z = y + 3$ and $z = -y - 3$, respectively. Since the normal vectors point outward (meaning away from the z-axis), the

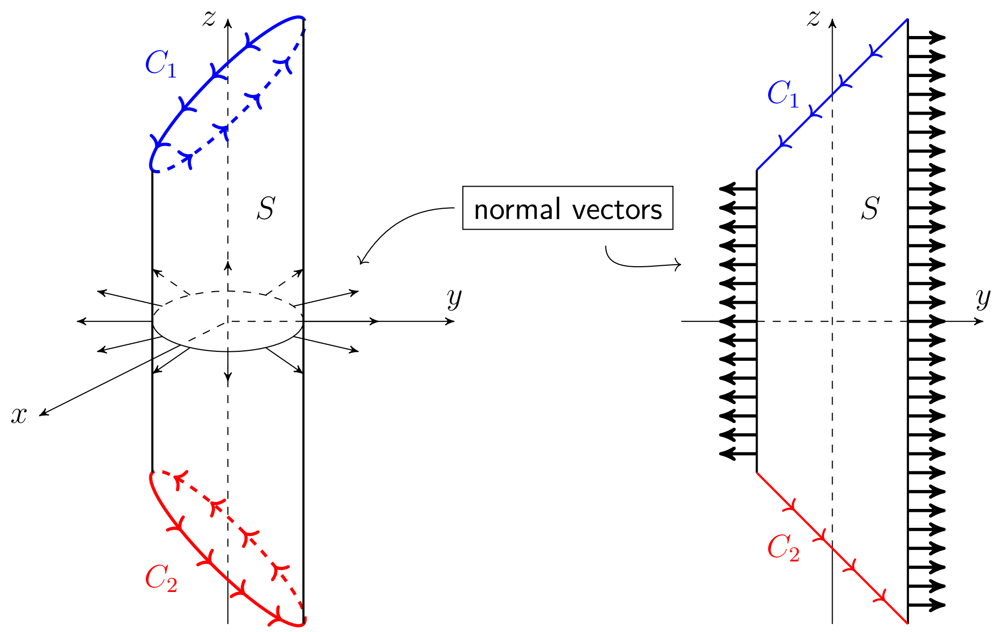

Figure 1: The surface S and its boundary $\partial S = C_1 \cup C_2$ (with positive orientation).

boundary becomes positively-oriented if the upper ellipse C_1 is traversed clockwise and the lower ellipse C_2 is traversed counter-clockwise when viewed from above.

b) Based on part a), the following vector functions can be used to represent the (oriented) boundary $\partial S = C_1 \cup C_2$:

$$\begin{aligned} C_1: &\qquad \vec{r}_1(t) = \cos t\,\vec{\imath} - \sin t\,\vec{\jmath} + (3 - \sin t)\,\vec{k}, \quad 0 \le t \le 2\pi, \\ C_2: &\qquad \vec{r}_2(t) = \cos t\,\vec{\imath} + \sin t\,\vec{\jmath} - (3 + \sin t)\,\vec{k}, \quad 0 \le t \le 2\pi. \end{aligned}$$

The circulation of $\vec{V}$ along the boundary $\partial S = C_1 \cup C_2$ is the sum of the line integrals of $\vec{V}$ along the two ellipses C_1 and C_2:

$$\int_{\partial S} \vec{V} \cdot d\vec{r} = \oint_{C_1} \vec{V} \cdot d\vec{r} + \oint_{C_2} \vec{V} \cdot d\vec{r}.$$

To compute these line integrals, we need the tangent vectors as well as the values $\vec{V}$ takes along the curves. Along C_1, we obtain

$$\vec{r}_1{}'(t) = -\sin t\,\vec{\imath} - \cos t\,\vec{\jmath} - \cos t\,\vec{k}$$

and

$$\begin{aligned} \vec{V}(\vec{r}_1(t)) &= \frac{-(-\sin t)}{\sqrt{(\cos t)^2 + (-\sin t)^2}}\,\vec{\imath} + \frac{\cos t}{\sqrt{(\cos t)^2 + (-\sin t)^2}}\,\vec{\jmath} + \big((\cos t)^2 + (-\sin t)^2\big)(3 - \sin t)\,\vec{k} \\ &= \sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + (3 - \sin t)\,\vec{k} \end{aligned}$$

and similarly, along C_2 we get

$$\vec{r}_2{}'(t) = -\sin t\,\vec{\imath} + \cos t\,\vec{\jmath} - \cos t\,\vec{k}$$

and

$$\begin{aligned} \vec{V}(\vec{r}_2(t)) &= \frac{-(\sin t)}{\sqrt{(\cos t)^2 + (\sin t)^2}}\,\vec{\imath} + \frac{\cos t}{\sqrt{(\cos t)^2 + (\sin t)^2}}\,\vec{\jmath} + \big((\cos t)^2 + (\sin t)^2\big)\big(-(3 + \sin t)\big)\,\vec{k} \\ &= -\sin t\,\vec{\imath} + \cos t\,\vec{\jmath} - (3 + \sin t)\,\vec{k}. \end{aligned}$$

Hence the line integral along the upper ellipse C_1 is

$$\oint_{C_1} \vec{V} \cdot d\vec{r} = \int_0^{2\pi} \vec{V}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\, dt$$
$$= \int_0^{2\pi} (\sin t\, \vec{\imath} + \cos t\, \vec{\jmath} + (3 - \sin t)\, \vec{k}) \cdot (-\sin t\, \vec{\imath} - \cos t\, \vec{\jmath} - \cos t\, \vec{k})\, dt$$
$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t - 3\cos t + \sin t \cos t)\, dt$$
$$= \int_0^{2\pi} (-1 - 3\cos t + \sin t \cos t)\, dt$$

whereas along the lower ellipse C_2 we get

$$\oint_{C_2} \vec{V} \cdot d\vec{r} = \int_0^{2\pi} \vec{V}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\, dt$$
$$= \int_0^{2\pi} (-\sin t\, \vec{\imath} + \cos t\, \vec{\jmath} - (3 + \sin t)\, \vec{k}) \cdot (-\sin t\, \vec{\imath} + \cos t\, \vec{\jmath} - \cos t\, \vec{k})\, dt$$
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t + 3\cos t + \sin t \cos t)\, dt$$
$$= \int_0^{2\pi} (1 + 3\cos t + \sin t \cos t)\, dt$$

When we add these two integrals the first two terms cancel leaving us with

$$\int_{\partial S} \vec{V} \cdot d\vec{r} = \int_0^{2\pi} 2\sin t \cos t\, dt = \left[\sin^2 t\right]_{t=0}^{2\pi} = \sin^2 2\pi - \sin^2 0 = 0 - 0 = 0.$$

Thus we see that the circulation of $\vec{V}$ along the boundary of S is zero.

c) Let us first compute the curl of $\vec{V}$:

$$\operatorname{curl}\vec{V} = \nabla\times\vec{V} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{\sqrt{x^2+y^2}} & \dfrac{x}{\sqrt{x^2+y^2}} & (x^2+y^2)\,z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left((x^2+y^2)\,z\right) - \frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right]\vec{\imath}$$

$$+ \left[\frac{\partial}{\partial z}\left(\frac{-y}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial x}\left((x^2+y^2)\,z\right)\right]\vec{\jmath}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{\sqrt{x^2+y^2}}\right)\right]\vec{k}$$

$$= \left[(2yz)-(0)\right]\vec{\imath} + \left[(0)-(2xz)\right]\vec{\jmath}$$

$$+ \left[\frac{\sqrt{x^2+y^2} - x\frac{1}{2}(x^2+y^2)^{-1/2}(2x)}{x^2+y^2} - \frac{-\sqrt{x^2+y^2} + y\frac{1}{2}(x^2+y^2)^{-1/2}(2y)}{x^2+y^2}\right]\vec{k}$$

$$= 2yz\,\vec{\imath} - 2xz\,\vec{\jmath} + \frac{1}{\sqrt{x^2+y^2}}\,\vec{k}.$$

The surface S is a piece of a cylinder so let us parameterize it in cylindrical coordinates using the vector function

$$\vec{r}(\theta,z) = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} + z\,\vec{k}, \qquad 0\le\theta\le 2\pi,\ -3-\sin\theta\le z\le 3+\sin\theta$$

remember that we have $-y-3\le z\le y+3$ and $y=\sin\theta$ on S hence the bounds for z. The derivatives of $\vec{r}(\theta,z)$ with respect to θ and z are

$$\vec{r}_\theta = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath}, \qquad \vec{r}_z = \vec{k}$$

and therefore we have

$$\vec{r}_\theta\times\vec{r}_z = (-\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath})\times\vec{k} = -\sin\theta\,(\vec{\imath}\times\vec{k}) + \cos\theta\,(\vec{\jmath}\times\vec{k}) = -\sin\theta\,(-\vec{\jmath}) + \cos\theta\,\vec{\imath}$$

that is

$$\vec{r}_\theta\times\vec{r}_z = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}.$$

The vector field $\operatorname{curl}\vec{V}$ takes the following values on S:

$$\operatorname{curl}\vec{V}(\vec{r}(\theta,z)) = 2z\sin\theta\,\vec{\imath} - 2z\cos\theta\,\vec{\jmath} + \frac{1}{\sqrt{\cos^2\theta+\sin^2\theta}}\,\vec{k} = 2z\sin\theta\,\vec{\imath} - 2z\cos\theta\,\vec{\jmath} + \vec{k}.$$

So the flux of $\operatorname{curl}\vec{V}$ across S is

$$\iint_S \operatorname{curl}\vec{V}\cdot d\vec{S} = \int_0^{2\pi}\int_{-3-\sin\theta}^{3+\sin\theta} \operatorname{curl}\vec{V}(\vec{r}(\theta,z))\cdot(\vec{r}_\theta\times\vec{r}_z)\,dz\,d\theta$$
$$= \int_0^{2\pi}\int_{-3-\sin\theta}^{3+\sin\theta} (2z\sin\theta\,\vec{\imath}-2z\cos\theta\,\vec{\jmath}+\vec{k})\cdot(\cos\theta\,\vec{\imath}+\sin\theta\,\vec{\jmath})\,dz\,d\theta$$
$$= \int_0^{2\pi}\int_{-3-\sin\theta}^{3+\sin\theta} 2z(\sin\theta\cos\theta-\cos\theta\sin\theta)\,dz\,d\theta$$
$$= \int_0^{2\pi}\int_{-3-\sin\theta}^{3+\sin\theta} 0\,dz\,d\theta = 0.$$

Thus we see that the flux of $\operatorname{curl}\vec{V}$ across S is zero.

4) Consider the vector field

$$\vec{F}(x,y,z) = xz\,\vec{\imath}+yz\,\vec{\jmath}+z^2\,\vec{k}$$

over the solid region $E=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le 4,\ z\ge\sqrt{x^2+y^2}\}$ and its outward-oriented boundary surface ∂E.

[2] a) Describe and sketch the region E and the surface ∂E (draw orientation).

Verify the Divergence Theorem by

[12] b) computing the flux of $\vec{F}$ across ∂E, that is $\iint_{\partial E}\vec{F}\cdot d\vec{S}$ and

[8] c) evaluating the triple integral of $\operatorname{div}\vec{F}$ over E, i.e. $\iiint_E \operatorname{div}\vec{F}\,dV$.

Solution. a)The inequality $x^2+y^2+z^2\le 4$ gives us the ball of radius 2 centred at the origin. The other inequality $z\ge\sqrt{x^2+y^2}$ yields a region above the circular cone whose axis is the positive z-axis with its apex at the origin and an apex angle of $\pi/2$. The solid region E is the intersection of the two regions, i.e. the piece of the ball that is above the cone (see Figure 2). Accordingly, the boundary of E is the union of a piece of sphere and a piece of cone. More precisely, let S_1 denote of the portion of the sphere of radius 2 centred at the origin for which we have the polar angle $0\le\phi\le\pi/4$. And let S_2 denote the piece of cone such that $x^2+y^2+z^2\le 4$. In summary, we have $\partial E = S_1\cup S_2$.

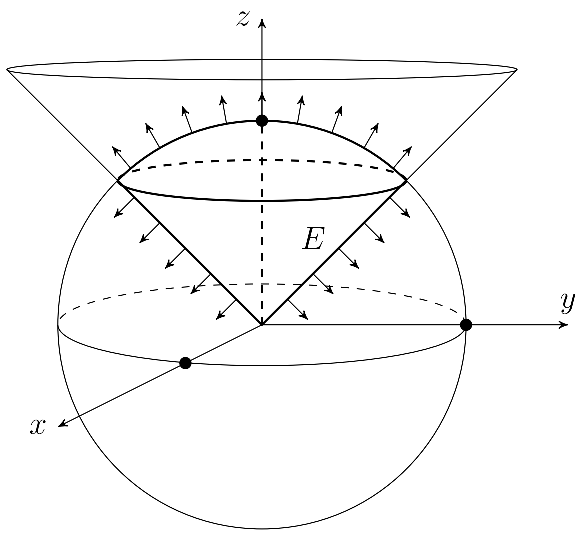

Figure 2: The solid region E (with outward-pointing normal vectors).

b) From part a), we deduce that S_1 is given by the vector function

$$S_1: \qquad \vec{r}_1(\phi,\theta) = 2\sin\phi\cos\theta\,\vec{\imath}+2\sin\phi\sin\theta\,\vec{\jmath}+2\cos\phi\,\vec{k},\quad 0\le\phi\le\pi/4,\ 0\le\theta\le 2\pi.$$

and the surface S_2 is given by the vector function

$$S_2: \qquad \vec{r}_2(\theta, z) = z\cos\theta\,\vec{\imath} + z\sin\theta\,\vec{\jmath} + z\,\vec{k}, \quad 0 \le \theta \le 2\pi,\ 0 \le z \le \sqrt{2}.$$

Note that the upper boundary for z was obtained by finding where the two surfaces $x^2+y^2+z^2=4$ and $z=\sqrt{x^2+y^2}$ intersect. There we have both equations satisfied, hence

$$4 = (x^2+y^2) + z^2 = (\sqrt{x^2+y^2})^2 + z^2 = (z)^2 + z^2 = 2z^2 \quad \Rightarrow \quad z^2 = 2,$$

which implies $z=\sqrt{2}$ since z is non-negative on the cone. The flux of $\vec{F}$ across $\partial E = S_1 \cup S_2$ is the sum of the surface integrals of $\vec{F}$ across S_1 and S_2:

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}.$$

To evaluate these surface integrals, we need to find (outward) normal vectors to the surfaces as well as the values of $\vec{F}$ on the surfaces. On S_1, we have

$$(\vec{r}_1)_\phi = 2\cos\phi\cos\theta\,\vec{\imath} + 2\cos\phi\sin\theta\,\vec{\jmath} - 2\sin\phi\,\vec{k}$$

and

$$(\vec{r}_1)_\theta = -2\sin\phi\sin\theta\,\vec{\imath} + 2\sin\phi\cos\theta\,\vec{\jmath}.$$

The cross product of these derivatives yields normal vectors to S_1:

$$\begin{aligned}
(\vec{r}_1)_\phi \times (\vec{r}_1)_\theta =& (2\cos\phi\cos\theta\,\vec{\imath} + 2\cos\phi\sin\theta\,\vec{\jmath} - 2\sin\phi\,\vec{k}) \times (-2\sin\phi\sin\theta\,\vec{\imath} + 2\sin\phi\cos\theta\,\vec{\jmath}) \\
=& (2\cos\phi\cos\theta)(2\sin\phi\cos\theta)(\vec{\imath}\times\vec{\jmath}) - (2\cos\phi\sin\theta)(2\sin\phi\sin\theta)(\vec{\jmath}\times\vec{\imath}) \\
& + (2\sin\phi)(2\sin\phi\sin\theta)(\vec{k}\times\vec{\imath}) - (2\sin\phi)(2\sin\phi\cos\theta)(\vec{k}\times\vec{\jmath}) \\
=& (4\sin\phi\cos\phi\cos^2\theta)(\vec{k}) - (4\sin\phi\cos\phi\sin^2\theta)(-\vec{k}) \\
& + (4\sin^2\phi\sin\theta)(\vec{\jmath}) - (4\sin^2\phi\cos\theta)(-\vec{\imath}) \\
=& 4\sin^2\phi\cos\theta\,\vec{\imath} + 4\sin^2\phi\sin\theta\,\vec{\jmath} + 4\sin\phi\cos\phi(\cos^2\theta+\sin^2\theta)\,\vec{k} \\
=& 4\sin^2\phi\cos\theta\,\vec{\imath} + 4\sin^2\phi\sin\theta\,\vec{\jmath} + 4\sin\phi\cos\phi\,\vec{k}.
\end{aligned}$$

As for the values of $\vec{F}$ on S_1, we get

$$\vec{F}(\vec{r}_1(\phi,\theta)) = 4\sin\phi\cos\phi\cos\theta\,\vec{\imath} + 4\sin\phi\cos\phi\sin\theta\,\vec{\jmath} + 4\cos^2\phi\,\vec{k}.$$

Hence the normal component of $\vec{F}$ is

$$\begin{aligned}
&\vec{F}(\vec{r}_1(\phi,\theta)) \cdot \big((\vec{r}_1)_\phi \times (\vec{r}_1)_\theta\big) \\
&= (4\sin\phi\cos\phi\cos\theta)(4\sin^2\phi\cos\theta) + (4\sin\phi\cos\phi\sin\theta)(4\sin^2\phi\sin\theta) + (4\cos^2\phi)(4\sin\phi\cos\phi) \\
&= 16\sin^3\phi\cos\phi(\cos^2\theta+\sin^2\theta) + 16\cos^3\phi\sin\phi \\
&= 16\sin^3\phi\cos\phi + 16\cos^3\phi\sin\phi.
\end{aligned}$$

The flux of $\vec{F}$ across S_1 is

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi/4} \vec{F}(\vec{r}_1(\phi,\theta)) \cdot ((\vec{r}_1)_\phi \times (\vec{r}_1)_\theta)\, d\phi\, d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi/4} (16\sin^3\phi\cos\phi + 16\cos^3\phi\sin\phi)\, d\phi\, d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} (16\sin^3\phi\cos\phi + 16\cos^3\phi\sin\phi)\, d\phi$$
$$= [\theta]_{\theta=0}^{\theta=2\pi} \left[4\sin^4\phi - 4\cos^4\phi\right]_{\phi=0}^{\phi=\pi/4}$$
$$= (2\pi)\left(4\sin^4\frac{\pi}{4} - 4\cos^4\frac{\pi}{4} - 4\sin^4 0 + 4\cos^4 0\right)$$
$$= (2\pi)\left(4\frac{1}{\sqrt{2}^4} - 4\frac{1}{\sqrt{2}^4} - 0 + 4\right) = (2\pi)(4) = 8\pi.$$

On S_2, we have the derivatives

$$(\vec{r}_2)_\theta = -z\sin\theta\,\vec{\imath} + z\cos\theta\,\vec{\jmath}, \qquad (\vec{r}_2)_z = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} + \vec{k}$$

and so we get the outward normal vectors

$$(\vec{r}_2)_\theta \times (\vec{r}_2)_z = (-z\sin\theta\,\vec{\imath} + z\cos\theta\,\vec{\jmath}) \times (\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} + \vec{k})$$
$$= (-z\sin^2\theta)(\vec{\imath}\times\vec{\jmath}) - z\sin\theta(\vec{\imath}\times\vec{k}) + (z\cos^2\theta)(\vec{\jmath}\times\vec{\imath}) + z\cos\theta(\vec{\jmath}\times\vec{k})$$
$$= (-z\sin^2\theta)(\vec{k}) - z\sin\theta(-\vec{\jmath}) + (z\cos^2\theta)(-\vec{k}) + z\cos\theta(\vec{\imath})$$
$$= z\cos\theta\,\vec{\imath} + z\sin\theta\,\vec{\jmath} - z(\sin^2\theta + \cos^2\theta)\,\vec{k}$$
$$= z\cos\theta\,\vec{\imath} + z\sin\theta\,\vec{\jmath} - z\,\vec{k}.$$

The vector field $\vec{F}$ takes the following values on S_2:

$$\vec{F}(\vec{r}_2(\theta,z)) = z^2\cos\theta\,\vec{\imath} + z^2\sin\theta\,\vec{\jmath} + z^2\,\vec{k}.$$

Its normal component is

$$\vec{F}(\vec{r}_2(\theta,z)) \cdot \big((\vec{r}_2)_\theta \times (\vec{r}_2)_z\big) = (z^2\cos\theta)(z\cos\theta) + (z^2\sin\theta)(z\sin\theta) + (z^2)(-z)$$
$$= z^3(\cos^2\theta + \sin^2\theta - 1) = 0$$

and hence the flux of $\vec{F}$ across S_2 is zero

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \int_0^{\sqrt{2}} \int_0^{2\pi} \vec{F}(\vec{r}_2(\theta,z)) \cdot ((\vec{r}_2)_\theta \times (\vec{r}_2)_z)\, d\theta\, dz = \int_0^{\sqrt{2}} \int_0^{2\pi} 0\, d\theta\, dz = 0.$$

Therefore the total flux of $\vec{F}$ across ∂E is 8π.

c) The divergence of $\vec{F}$ is

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(z^2) = z + z + 2z = 4z.$$

The solid E can be expressed in terms of spherical coordinates as follows

$$E: \qquad 0 \le \rho \le 2, \quad 0 \le \phi \le \frac{\pi}{4}, \quad 0 \le \theta \le 2\pi.$$

Therefore the triple integral of $\operatorname{div} \vec{F}$ over E is

$$\iiint\limits_E \operatorname{div} \vec{F}\, dV = \iiint\limits_E 4z\, dV = \int_0^2 \int_0^{\pi/4} \int_0^{2\pi} (4\rho\cos\phi)(\rho^2 \sin\phi)\, d\theta\, d\phi\, d\rho$$
$$= \int_0^2 4\rho^3\, d\rho \int_0^{\pi/4} \sin\phi\cos\phi\, d\phi \int_0^{2\pi} d\theta = \left[\rho^4\right]_{\rho=0}^{\rho=2} \left[\frac{\sin^2\phi}{2}\right]_{\phi=0}^{\phi=\pi/4} [\theta]_{\theta=0}^{\theta=2\pi}$$
$$= (2^4)\left(\frac{\sin^2\frac{\pi}{4}}{2} - \frac{\sin^2 0}{2}\right)(2\pi) = 16\left(\frac{\frac{1}{\sqrt{2}^2}}{2}\right)(2\pi) = 16\left(\frac{1}{4}\right)(2\pi) = 8\pi.$$